PERIOD–II

M12CH6

TOPIC **6**

Interpretation of Linear and Quadratic Graphs

6.1. GRAPHICAL METHOD TO SOLVE A PAIR OF LINEAR EQUATIONS IN TWO VARIABLES

Two linear equations in two variables form a system of linear equations. A solution to a system of linear equations is an ordered pair which satisfies both equations in the system.

For example, the ordered pair (3, 4) satisfies the system of equations

$$2x - 3y + 6 = 0$$

and $$2x - y - 2 = 0$$

since $$2 \times 3 - 3 \times 4 + 6 = 0$$

i.e., $$6 - 12 + 6 = 0$$

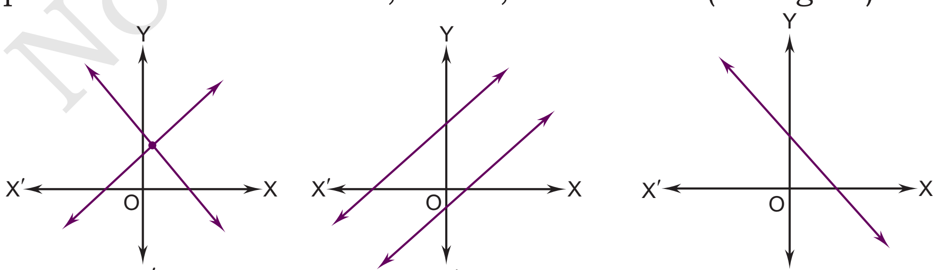
and $$2 \times 3 - 4 - 2 = 0 \quad \text{i.e.,} \quad 6 - 6 = 0$$

both are true statements.

Thus, $x = 3$ and $y = 4$ or the ordered pair (3, 4) is a solution of the given system of equations. The solution set to the system is {(3, 4)}.

The solution can be obtained by graphing both equations. The coordinates of the point of intersection give the solution of the system.

If the lines intersect, they will intersect in only one point giving a unique solution to the system (see figure). If the lines are parallel, there is no point of intersection and, hence, no solution (see figure).



Unique solution (intersecting lines) | No solution (parallel lines) | Infinitely many solutions (coincident lines)

If the lines are coincident, i.e., same line for both equations, then every point on the line is a common point and, hence, there are infinitely many solutions (see figure).

We shall concentrate on system having a unique solution.

Example 1. *Find the solution set of the following system of equations graphically:*

$$2x + 5y = 10 \quad \text{and} \quad x = -5$$

Solution. Given equations are:

$$2x + 5y = 10 \quad \ldots(1) \qquad x = -5 \quad \ldots(2)$$

From (1), $y = \frac{10 - 2r}{5}$

Table of values for (1)

x	0	5
y	2	0

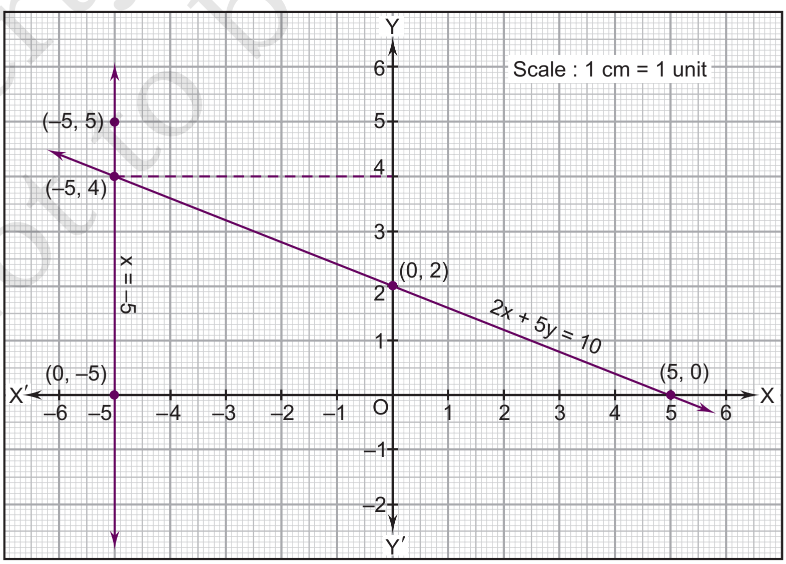
Table of values for (2)

x	–5	–5
y	0	5

Plot the ordered pairs (0, 2) and (5, 0). Join and produce both ways. This is the graph of equation (1).

Plot the ordered pairs (–5, 0) and (–5, 5). Join and produce both ways. This vertical line is the graph of equation (2).

The two lines intersect at a unique point (–5, 4) (Figure). Therefore, the ordered pair (–5, 4), i.e., $x = -5$, $y = 4$ is the solution of the system. The solution set $S = \{(-5, 4)\}$ is a singleton set.



6.2. NON-GRAPHICAL METHODS FOR SOLVING A SYSTEM OF LINEAR EQUATIONS IN TWO VARIABLES

(a) Elimination Method

Procedure:

(i) If necessary, re-write the equations so that the constant term in both equations is on the same side, left or right.

(ii) Multiply one or both equations, if necessary, by appropriate non-zero numbers so that addition or subtraction eliminates one variable. Now we have an equation in one variable, x or y.

(iii) Solve the resulting single variable equation for the variable involved.

(iv) Substitute the value obtained in step (iii) in either of the two original equations and solve for the other variable.

Example 2. *Solve the following system of equations by elimination method:*

$$4x + 3y - 14 = 0, \quad 9x - 5y = 55.$$

Solution. The given equations are

$$4x + 3y = 14 \quad ...(1) \quad \text{and} \quad 9x - 5y = 55 \quad ...(2)$$

(Re-write equation (1) so that the constant term is on same side in both equations.)

Let us eliminate *y*

Co-efficients of y in the two equations are 3 and 5 (numerically). LCM of 3 and 5 is 15. To make both co-efficients 15 (numerically), we multiply equation (1) by 5 and equation (2) by 3, we get

$$20x + 15y = 70...(3) \quad \text{and} \quad 27x - 15y = 165 \quad ...(4)$$

Adding (3) and (4), we get

$$47x = 235 \quad [y \text{ eliminated}]$$

$$\Rightarrow \quad x = \frac{5 \times 47}{47} = 5$$

Replacing x by 5 in equation (1), we get

$$4 \times 5 + 3y = 14 \quad \Rightarrow \quad 3y = 14 - 20$$

$$\Rightarrow \quad 3y = -6 \quad \Rightarrow \quad y = \frac{-6}{3} = -2$$

Therefore, the solution is $x = 5$, $y = -2$.

Hence, the solution set is $S = \{(5, -2)\}$.

(b) Substitution Method

Procedure:

(i) Solve one of the given equations for one variable in terms of the other, i.e., x in terms of y or y in terms of x.

(ii) Substitute this value of the variable in the other equation. This results into a single variable linear equation.

(iii) Solve the equation of step (ii)

(iv) Substitute this value in the expression obtained in step (i) to get the value of other variable.

Example 3. *Solve the following system of equations by substitution method:*

$$14x - 3y = 54, \quad 21x - 8y = 95.$$

Solution. The given equations are

$$14x - 3y = 54 \qquad \ldots(1)$$

and

$$21x - 8y = 95 \qquad \ldots(2)$$

Let us find *x* in terms of *y* from equation (1)

(We can find the same from equation (2) also.)

From (1), $14x = 54 + 3y$

or $x = \dfrac{54 + 3y}{14}$ ...(3)

Substituting this value of x in equation (2), we get

$$21\left(\frac{54 + 3y}{14}\right) - 8y = 95 \quad \Rightarrow \quad 3 \times 7\left(\frac{54 + 3y}{2 \times 7}\right) - 8y = 95$$

$$\Rightarrow \quad 3\left(\frac{54 + 3y}{2}\right) - 8y = 95$$

Multiplying both sides by 2, we get

$$3\,(54 + 3y) - 2\,(8y) = 2 \times 95$$

$$\Rightarrow \quad 162 + 9y - 16y = 190$$

$$\Rightarrow \quad -7y = 190 - 162$$

$$\Rightarrow \quad -7y = 28$$

$$\Rightarrow \quad y = \frac{4 \times 7}{-7} = -4$$

Substituting this value of y in equation (3), we get

$$x = \frac{54 + 3(-4)}{14} = \frac{54 - 12}{14} = \frac{42}{14} = 3$$

Therefore, the solution is $x = 3$, $y = -4$

Hence the solution set is $S = \{(3, -4)\}$.

6.3. GRAPHICAL METHOD TO SOLVE QUADRATIC EQUATION

An alternative method for solving a quadratic equation graphically is to find the intersection of the curve $y = x^2$ with a particular line.

Example 4. *Draw the graph of $y = x^2$ for $-3 \leq x \leq 3$.*

Use the graph to solve $x^2 + 2x - 3 = 0$.

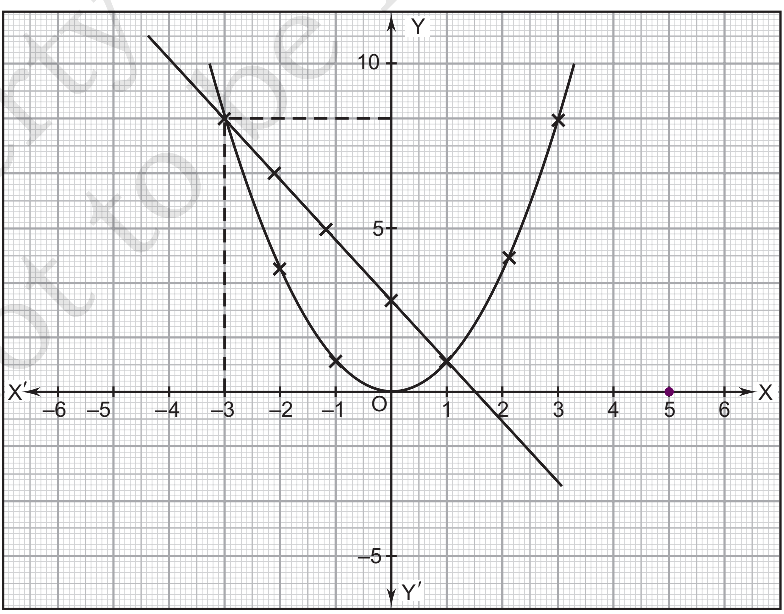
The equation $x^2 + 2x - 3 = 0$ is the same as $x^2 = -2x + 3$.

The table of values is:

x	–3	–2	–1	0	1	2	3
x^2	9	4	1	0	1	4	9
$-2x + 3$	9	7	5	3	1	–1	–3

Draw the graph of $y = x^2$ and the line $y = -2x + 3$ on the same coordinate axes.

Solution.



6.4. QUADRATIC FUNCTIONS AND EQUATIONS

Solving Quadratic Equations by Factorization

(a) If a, b and c are real numbers with $a \neq 0$, then the function

$$f(x) = ax^2 + bx + c$$

is called a **quadratic function**.

For example, $3x^2 - 5x + 2$, $\sqrt{2}x^2 - 2.5x + 1$ are quadratic functions.

Since, $f(x)$ represents a real number for any real value of x, the domain of a quadratic function is R, the set of all real numbers. The range of a quadratic function depends on the values of a, b and c.

(b) If a, b, and c are real numbers with $a \neq 0$, then the equation

$$ax^2 + bx + c = 0 \qquad \ldots(1)$$

is called a **quadratic equation** in **standard form**.

Clearly, if $f(x)$ is a quadratic function in x, then $f(x) = 0$ is a quadratic equation in x.

The values of x which satisfy equation (1) are called the **solutions** or **roots** or **truth values** of equation (1).

A real number α is said to satisfy equation (1) if $a\alpha^2 + b\alpha + c = 0$, i.e., if on replacing x by α in the left hand side, we get the right hand side (0).

Every quadratic equation has exactly two real roots if $b^2 - 4ac \geq 0$. The expression $b^2 - 4ac$ is called the **discriminant** of the quadratic equation.

If $b^2 - 4ac < 0$, then the quadratic equation has no real roots.

If $b^2 - 4ac > 0$, then the two roots are real and unequal.

If $b^2 - 4ac = 0$, then the two roots are real and equal.

The set of all truth values or solutions of a quadratic equation is called the truth set or the solution set of the quadratic equation. It is denoted by T or S. Thus,

$$T = \phi \quad \text{if } b^2 - 4ac < 0$$
$$T = \{\alpha\} \quad \text{if } b^2 - 4ac = 0$$
$$T = \{\alpha, \beta\} \quad \text{if } b^2 - 4ac > 0$$

A quadratic equation can be solved by a variety of methods

(i) by factorization

(ii) by quadratic formula

(iii) by the method of completing a square

(iv) by graph

Here, we discuss the first method only and assume that $b^2 - 4ac \geq 0$ so that the quadratic equation has two real roots.

Method

I. Write the given equation in the standard form $ax^2 + bx + c = 0$, by transposing all terms to the left hand side.

If a is negative, then multiply both sides by (–1), i.e., change the sign throughout so that a is positive.

II. Factorize the left hand side.

(i) If $b = 0$, use $X^2 - Y^2 = (X + Y)(X - Y)$

(ii) If $c = 0$, then $ax^2 + bx = 0 \Rightarrow x(ax + b) = 0$

(iii) If a, b and c are all non-zero, then find two numbers α and β such that $\alpha + \beta = b$ and $\alpha\beta = ac$.

In the given equation, replace b by $(\alpha + \beta)$, apply distributive property and group in pairs. Thus,

$$ax^2 + bx + c = 0$$

$$\Rightarrow \quad (px + q)(rx + s) = 0$$

III. Apply zero-product principle, which states that if the product of two algebraic expressions is zero, then at least one of the factors is equal to zero. Thus,

$$(px + q)(rx + s) = 0$$

$$\Rightarrow \quad px + q = 0 \quad \text{or} \quad rx + s = 0$$

$$\Rightarrow \quad x = -\frac{q}{p} \quad \text{or} \quad x = -\frac{s}{r}$$

IV. Check the solution in the original equation.

Hence, the truth set is $T = \left\{-\frac{q}{p}, -\frac{s}{r}\right\}$.

Example 5. *Solve by factorization:*

$$3x^2 + 7x = 0.$$

Solution. The given equation is

$$3x^2 + 7x = 0 \qquad \ldots(1)$$

(here $c = 0$)

or $\quad x(3x + 7) = 0$

By zero-product principle, we have

$$x = 0 \quad \text{or} \quad 3x + 7 = 0$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad 3x = -7$$

$$\Rightarrow \quad x = 0 \quad \text{or} \quad x = -\frac{7}{3}$$

Example 6. *Solve by factorization:*

$$2x^2 + 10 = 9x.$$

Solution. The given equation is

$$2x^2 + 10 = 9x \qquad \ldots(1)$$

Transposing all terms to left hand side

$$2x^2 - 9x + 10 = 0$$

Here $a = 2$, $b = -9$, $c = 10$

The two numbers whose sum is $b = -9$ and product is $ac = 2 \times 10 = 20$ are -5 and -4.

Replacing -9 by $-5 - 4$, we get

$$2x^2 - 5x - 4x + 10 = 0$$

$$\Rightarrow \quad x(2x - 5) - 2(2x - 5) = 0$$

$$\Rightarrow \quad (2x - 5)(x - 2) = 0$$

By zero-product principle

$$2x - 5 = 0 \quad \text{or} \quad x - 2 = 0$$

$$\Rightarrow \quad 2x = 5 \quad \text{or} \quad x = 2$$

$$\Rightarrow \quad x = \frac{5}{2} \quad \text{or} \quad x = 2$$

6.5. GRAPHICAL SOLUTION OF QUADRATIC EQUATIONS

The standard form of a quadratic equation is

$$ax^2 + bx + c = 0 \qquad \ldots(1)$$

where a, b and c are real numbers and $a \neq 0$.

If we write $$f(x) = ax^2 + bx + c \qquad \ldots(2)$$

then (2) is a quadratic function.

To solve the quadratic equation (1)graphically, we draw the graph of quadratic function (2), i.e., $y = ax^2 + bx + c$ where $y = f(x)$. The values of x for which $y = 0$, i.e., $ax^2 + bx + c = 0$ are the solutions of equation (1). On the graph of $y = f(x)$, $y = 0$ where the graph intersects x-axis.

If the graph

(i) does not intersect x-axis anywhere, then equation (1) has no solution or root or truth value.

The truth set T = ϕ

(ii) touches x-axis at the point $(\alpha, 0)$, the $x = \alpha$ is the twice repeated solution of equation (1).

The truth set T = $\{\alpha\}$

(iii) intersects x-axis at two distinct points $(\alpha, 0)$ and $(\beta, 0)$, then $x = \alpha$ and $x = \beta$ are the two solutions of equation (1).

The truth set T = $\{\alpha, \beta\}$.

Graph of $f(x) = ax^2 + bx + c$

Let $y = f(x)$, then $y = ax^2 + bx + c$

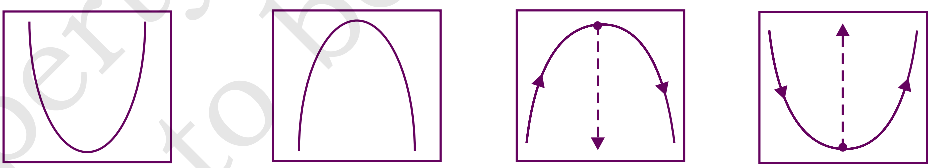
Put $x = x_1, x_2, \ldots, x_7$ and find the corresponding values of y. Display the values in the form of a table, called the table of values. Plot the seven ordered pairs (x, y) and pass a smooth curve through them. This curve is called a Parabola.

The graph of

$$y = ax^2 + bx + c,\ a \neq 0 \qquad \ldots(1)$$

is always a parabola.

When $a > 0$, the parabola opens upwards and is, therefore, called an upward parabola.



When $a < 0$, the parabola opens downwards and is, therefore, called Downward Parabola.

An important point in the graph of a parabola is the point where the graph takes a turn. This point is called the **vertex** of the parabola.

In an upward parabola, vertex is the lowest point. On the left of the vertex, the graph is falling and on the right of the vertex, the graph is rising.

In a downward parabola the vertex is the highest point. On the left of the vertex, the graph is rising and on the right of the vertex, the graph is falling.

The x-coorrdinate of the vertex is given by $-\frac{b}{2a}$. The y-coordinate can be obtained by putting $x = -\frac{b}{2a}$ in equation (1).

The line drawn vertically upwards through the vertex of an upward parabola is called the **axis** of the parabola. It is the line of symmetry for the upward parabola. Folding the graph sheet about this line, the two halves of the parabola coincide.

The line drawn vertically downwards through the vertex of a downward parabola is called the axis of the parabola. It is the line of symmetry for the downward parabola. Folding the graph sheet about this line, the two halves of the parabola coincide.

Steps for solving $ax^2 + bx + c = 0$, $a \neq 0$, graphically.

1. Let $y = ax^2 + bx + c$

2. Check the sign of a.

$a > 0 \Rightarrow$ upward parabola

$a < 0 \Rightarrow$ downward parabola

3. Find $-\frac{b}{2a}$. Put $x = -\frac{b}{2a}$ in equation (1) and find y. This gives the vertex of parabola, say (x_0, y_0).

4. Form the table of values. Give six integral values of x, three less than x_0 and three more than x_0. This gives seven ordered pairs (x, y).

5. Plot the seven ordered pairs.

6. Passasmoothcurvethroughalltheplottedpoints.Thisisthegraphof $f(x) = ax^2 + bx + c$.

7. Mark the points where the graph intersects x-axis and find the ordered pairs corresponding to them.

8. The x-coordinates of points in step (7) are the required solutions or truth values or roots of the quadratic equation $ax^2 + bx + c = 0$.

Example 7. *Draw the graph of $f(x) = x^2 - 2x + 2$ and hence solve the equation $x^2 - 2x + 2 = 0$.*

Solution. Let $y = x^2 - 2x + 2$ ...(1)

Here $a = 1$, $b = -2$, $c = 2$

Since $a > 0$, the graph of (1) is an upward parabola.

$$-\frac{b}{2a} = -\frac{-2}{2\times 1} = 1$$

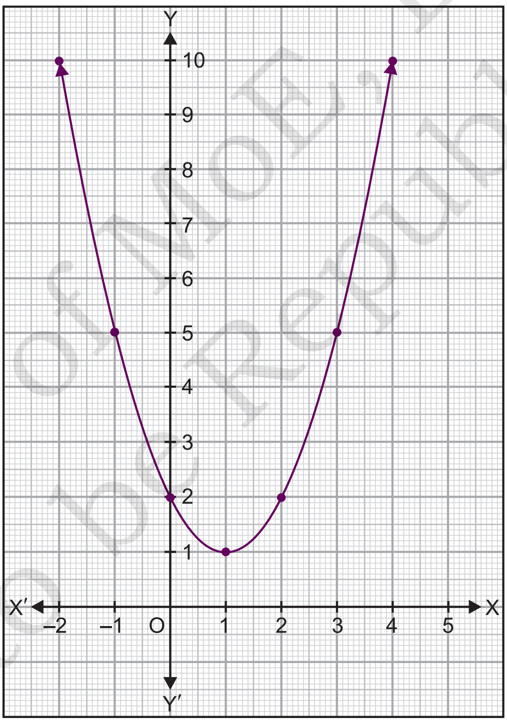
Putting $x = 1$ in (1), $y = 1^2 - 2 \times 1 + 2 = 1$

$\Rightarrow$ (1, 1) is the vertex.

Table of values $y = x^2 - 2x + 2$

x	–2	–1	0	1	2	3	4
y	10	5	2	1	2	5	10

Plot the seven ordered pairs. Pass a smooth curve through all the plotted ordered pairs. This is the graph of $y = x^2 - 2x + 2$.



This is an upward parabola. The arrows at the two ends indicate that the parabola extends indefinitely in both directions. Since the graph does not intersect x-axis anywhere, the quadratic equation $x^2 - 2x + 2 = 0$ has no truth value and the truth set $T = \phi$.

Example 8. *Draw the graph of $f(x) = -4x^2 - 12x - 9$ and, hence, solve the equation $4x^2 + 12x + 9 = 0$.*

Solution. Let $y = -4x^2 - 12x - 9$ ...(1)

Here $a = -4$, $b = -12$, $c = -9$

Since $a < 0$, the graph of (1) is a downward parabola.

$$-\frac{b}{2a} = -\frac{-12}{2(-4)} = -\frac{3}{2}$$

Putting $x = -\frac{3}{2}$ in (1),

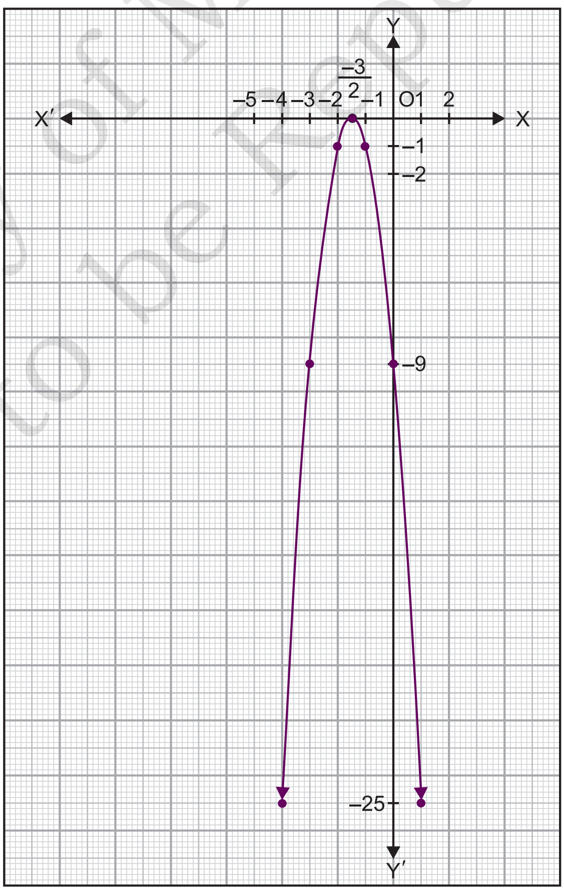
$$y = -4\left(-\frac{3}{2}\right)^2 - 12\left(-\frac{3}{2}\right) - 9$$

$$= -4\left(\frac{9}{4}\right) + 18 - 9 = -9 + 9 = 0$$

$\Rightarrow \left(-\frac{3}{2}, 0\right)$ is the vertex or turning point of (1).

Table of values for $y = -4x^2 - 12x - 9$

x	–4	–3	–2	–3/2	–1	0	1
y	–25	–9	–1	0	–1	–9	–25



Plot the seven ordered pairs. Pass a smooth curve through all the plotted ordered pairs. This is the graph of $y = -4x^2 - 12x - 9$. This is a downward parabola. The arrow at the two ends indicates that the parabola extends indefinitely in both directions.

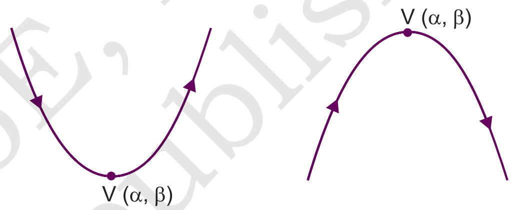
Since the graph touches x-axis at $\left(-\frac{3}{2}, 0\right)$, $x = -\frac{3}{2}$ is the only solution of the quadratic equation

$$-4x^2 - 12x - 9 = 0 \quad \text{or} \quad 4x^2 + 12x + 9 = 0$$

Hence, the truth set $T = \{-3/2\}$

6.6. INCREASING/DECREASING VALUES OF QUADRATIC GRAPHS

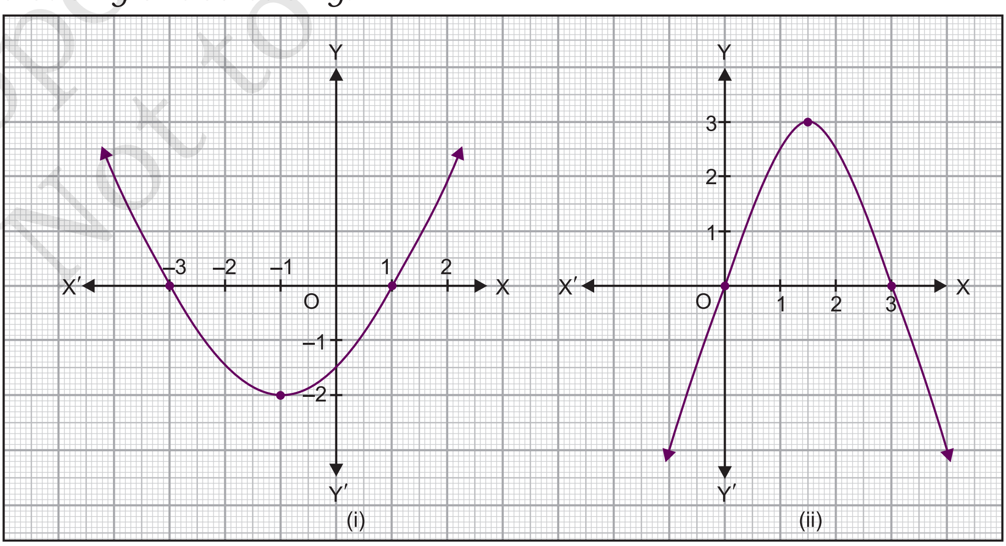
We know that the graph of a quadratic function is a parabola, upward or downward. Vertex of a parabola is a turning point. Let $V(\alpha, \beta)$ be the vertex of a parabola.



In an upward parabola, the graph is falling on the left of V and rising on the right of V. Thus, $y = f(x)$ is decreasing for $-\infty < x < \alpha$ and increasing for $\alpha < x < \infty$.

In a downward parabola, the graph is rising on the left of V and falling on the right of V. Thus, $y = f(x)$ is increasing for $-\infty < x < \alpha$ and decreasing for $\alpha < x < \infty$.

Example 9. *Find the range of values of x for which the following graphs are increasing or decreasing.*



Solution. In figure (i), the vertex is $V(-1, -2)$ so that $\alpha = -1$. The parabola opens upwards. The graph is falling on the left of V and rising on the right of V.

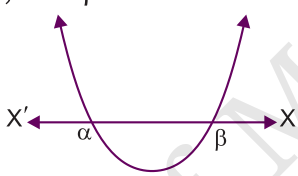
Thus, $y = f(x)$ is decreasing for $-\infty < x < -1$ and increasing for $-1 < x < \infty$.

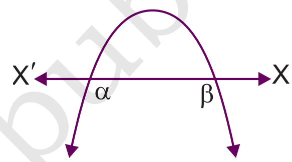
In Figure (ii), the vertex is $V(1.5, 3)$ so that $\alpha = 1.5$. The parabola opens downwards. The graph is rising on the left of V and falling on the right of V.

Thus, $y = f(x)$ is increasing for $-\infty < x < 1.5$ and decreasing for $1.5 < x < \infty$.

6.7. POSITIVE/NEGATIVE VALUES OF A QUADRATIC GRAPH

(a) Let a quadratic graph intersect x-axis at points whose x-coordinates are α and β, $\alpha < \beta$.





If the graph is an upward parabola, figure, then the graph is

(i) above x-axis, i.e., y is positive for $x < \alpha$ and $x > \beta$.

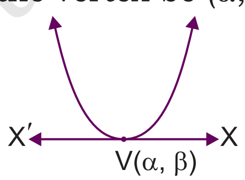
(ii) below x-axis, i.e., y is negative for $\alpha < x < \beta$.

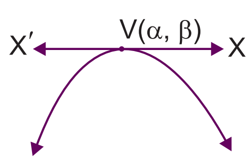
If the graph is a downward parabola, figure, then the graph is

(i) above x-axis, i.e., y is positive for $\alpha < x < \beta$

(ii) below x-axis, i.e., y is negative for $x < \alpha$ and $x > \beta$.

(b) Suppose a quadratic graph touches x-axis, then the point of contact is the vertex. Let the vertex be (α, β).

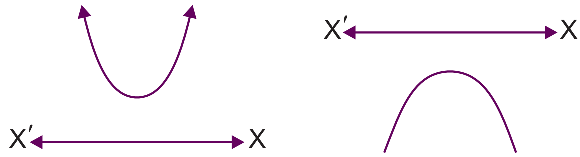




If the graph is an upward parabola, Fig. 3, then the graph is above x-axis except at the vertex. Thus y is positive for all real values of x except 'α', i.e., for $R - \{\alpha\}$.

If the graph is a downward parabola, figure, then the graph is below x-axis except at the vertex. Thus, y is negative for all real values of x except 'α', i.e., for $R - \{\alpha\}$.

(c) Suppose a quadratic graph does not intersect x-axis anywhere.



If the graph is an upward parabola, figure, then the graph is entirely above x-axis. Thus, y is positive for all $x \in R$, i.e., for $-\infty < x < \infty$.

If the graph is a downward parabola, figure, then the graph is entirely below x-axis. Thus, y is negative for all $x \in R$, i.e., $-\infty < x < \infty$.

Example 10. *Find the range of values of x for which $y = x^2 - 4x + 3$ is positive or negative.*

Solution. Given $y = x^2 - 4x + 3$

Here, $a = 1,\ b = -4,\ c = 3$

Since $a > 0$, the parabola opens upwards.

The parabola intersects x-axis where $y = 0$

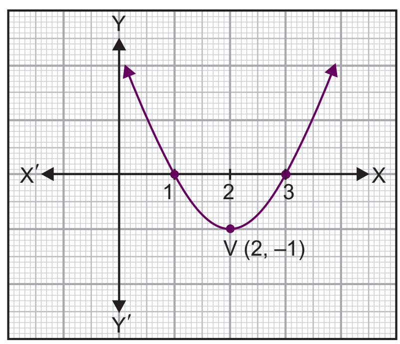
i.e., $x^2 - 4x + 3 = 0$ or $(x - 1)(x - 3) = 0$

$\Rightarrow$ $x = 1$ or 3

$$-\frac{b}{2a} = -\frac{-4}{2 \times 1} = 2$$

Putting $x = 2$, $y = 2^2 - 4 \times 2 + 3 = -1$

$\Rightarrow$ Vertex is $(2, -1)$.



Therefore, y is positive, i.e., the graphs is above x-axis when $x < 1$ or $x > 3$.

y is negative, i.e., the graph is below x-axis when $1 < x < 3$.

EXERCISE

1. Find the solution set of the following system of equations graphically:

$$2x - y - 1 = 0 \quad \text{and} \quad x - 2y + 1 = 0.$$

2. Solve the factorization $4x^2 - 49 = 0$.

3. Solve the factorization $9x^2 - 30x + 25 = 0$.

4. Solve $x^2 + 2x - 8 = 0$ graphically.